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# THE DETERMINANT OF A ROW- FACTORIZATION MATRIX IN A NUMERICAL SEMIGROUP (Researches on isometries from various viewpoints)

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# THE DETERMINANT OF A ROW-FACTORIZATION MATRIX IN A NUMERICAL SEMIGROUP

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## 1. NOTATIONS AND DEFINITIONS

In this paper, we study the determinant of row-factorization matrix in a numerical semigroup. Row-factorization matrices, in short RF-matrices, for pseudo-Frobenius numbers  $f$  in a numerical semigroup  $S$  are defined by Moscariello in [5], to prove the type of almost symmetric semigroups generated by four elements is less than or equal to three. Their determinants, in general, is multiples of  $f$ . If its absolute value is  $f$ , then we get a basis of the kernel space defined by  $S$ , from the RF-matrix. Then we can also get a generating system of a defining ideal of the semigroup associated with  $S$ . Hence, it is important to investigate the determinants of RF-matrices.

First, we give notations and definitions. Let  $\mathbb{Z}$  be the ring of integers, and  $\mathbb{N}$  the set of non negative integers. Let  $S$  be a non empty subset in  $\mathbb{N}$ . We say that  $S$  is a *semigroup* in  $\mathbb{N}$ , if

- (1)  $0 \in S$ ,
- (2)  $a + b \in S$ , if  $a, b \in S$ .

Let  $S$  be a semigroup in  $\mathbb{N}$  and  $n_1, \dots, n_s \in \mathbb{N}$ . We say that  $S$  is *generated* by  $n_1, \dots, n_s$  if

$$S = \{a_1 n_1 + \dots + a_s n_s : a_1, \dots, a_s \in \mathbb{N}\}.$$

We also say that  $S$  is *minimally* generated by  $n_1, \dots, n_s$ , if any proper subset of  $\{n_1, \dots, n_s\}$  does not generate  $S$ . Then we denote  $S$  by  $\langle n_1, \dots, n_s \rangle$  and call  $s$  the *embedding dimension* of  $S$ .

If  $\mathbb{N} - S$  is finite, we say that  $S$  is *numerical*. We note that  $S$  is numerical if and only if the general common divisor of  $n_1, \dots, n_s$  is one.

**Example 1.**

$$\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \dots\}$$

is a numerical semigroup generated by 3 and 5.

From now, all semigroups are assumed to be numerical semigroups in  $\mathbb{N}$ . Let  $S$  be a semigroup. The number

$$F(S) = \max\{a \in \mathbb{Z} : a \notin S\}$$

is called the *Frobenius number* of  $S$ . We also define

$$PF(S) = \{a \in \mathbb{Z} : a + x \in S \text{ if } x \in S \text{ and } x \neq 0\}$$

and an element in  $PF(S)$  is called a *pseudo-Frobenius number*. Obviously,  $F(S) \in PF(S)$ . We say that the number of  $PF(S)$  is the *type* of  $S$ , denoted by  $t(S)$ . For  $d \in S$ , We define the *Apery set*  $Ap(S, d)$  as follows:

$$Ap(S, d) = \{x \in S : x - d \notin S\}.$$

Note  $|Ap(S, d)| = d$  and, for any  $a \in \{0, 1, \dots, d-1\}$ , there is  $x \in Ap(S, d)$  with  $a \equiv x \pmod{d}$ .

Next, we define RF-matrices. Let  $S = \langle n_1, \dots, n_s \rangle$  be a semigroup and  $f \in \mathbb{Z} - S$ . For each  $i$ , there is  $a_{ii} < 0$  with  $f - a_{ii}n_i \in Ap(S, n_i)$ . Then there are  $a_{ij} \geq 0$  for  $j \neq i$  satisfying  $f - a_{ii}n_i = \sum_{j \neq i} a_{ij}n_j$ . We say that the matrix  $RF(f) = (a_{ij})$  is an RF-matrix (row-factorization matrix) for  $f$  in  $S$ . We denote it by  $RF(f)$ .

**Example 2.** (Examples of RF-matrices)

(1) Let  $S = \langle 3, 4, 5 \rangle$  and  $f = 2 \notin S$ . Then

$$RF(2) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

(2) Let  $S = \langle 4, 5, 6 \rangle$  and  $f = 7 \notin S$ . Then

$$RF(7) = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}.$$

From above example, it follows that an RF-matrix for pseudo-Frobenius number is not unique in general.

## 2. THE DETERMINANTS OF RF-MATRICES

In this section, we consider the following question: Let  $S$  be a numerical semigroup with embedding dimension  $s$  and  $f \in PF(S)$ . Then, does the equation

$$(*) \quad \det RF(f) = (-1)^{s+1} f$$

hold?

**Theorem 1.** If  $s = 2$  or  $3$ , then  $(*)$  holds.

*Proof.* Assume  $s = 2$  and let  $S = \langle n_1, n_2 \rangle$ . Then  $F(S) = n_1 n_2 - n_1 - n_2$  is a unique pseudo-Frobenius number and

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & n_1 - 1 \\ n_2 - 1 & -1 \end{pmatrix},$$

thus  $\det \text{RF}(F(S)) = -F(S)$ .

Assume  $s = 3$  and let  $S = \langle n_1, n_2, n_3 \rangle$ . If  $t(S) = 1$ , then we may assume  $dn_3 \in \langle n_1, n_2 \rangle$  where  $d = \gcd(n_1, n_2)$ . Then  $F(S) = n_1 n_2 / d - n_1 - n_2 + (d - 1)n_3$  is a unique pseudo-Frobenius number and

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & n_1/d - 1 & d - 1 \\ n_2/d - 1 & -1 & d - 1 \\ a_{31} & a_{32} & -1 \end{pmatrix},$$

where  $dn_3 = (a_{31} + 1 - n_2/d)n_1 + (a_{32} + 1)n_2$  or  $(a_{31} + 1)n_1 + (a_{32} + 1 - n_1/d)n_2$ . Then  $\det \text{RF}(F(S)) = F(S)$ .

The rest case is that of  $s = 3$  and  $t(S) = 2$ . Let  $\text{PF}(S) = \{f_1, f_2\}$  and put  $\text{RF}(f_1) = (a_{ij})$  and  $\text{RF}(f_2) = (b_{ij})$ . By classical result, they are unique and we may assume

$$\begin{aligned} a_{12} &= b_{32} = a_{32} + b_{12} + 1, \\ a_{23} &= b_{13} = a_{13} + b_{23} + 1, \\ a_{31} &= b_{21} = a_{21} + b_{31} + 1 \end{aligned}$$

Since

$$\begin{aligned} n_1 &= (a_{12} + 1)(a_{13} + 1) + (b_{12} + 1)(b_{23} + 1), \\ n_2 &= (a_{23} + 1)(a_{21} + 1) + (b_{23} + 1)(b_{31} + 1), \\ n_3 &= (a_{31} + 1)(a_{32} + 1) + (b_{31} + 1)(b_{12} + 1), \end{aligned}$$

we have  $\det \text{RF}(f_i) = f_i$  for  $i = 1, 2$ . □

**Definition.** Let  $S_1, S_2$  be numerical semigroups and  $d_1 \in S_2$  and  $d_2 \in S_1$ . If  $d_1$  and  $d_2$  are coprime, then

$$S = d_1 S_1 + d_2 S_2 = \{d_1 x + d_2 y : x \in S_1, y \in S_2\}$$

is a numerical semigroup. We say that  $S$  is *glued* by  $S_1$  and  $S_2$ .

**Definition.** We say that  $S$  is *completely glued* if one of the following is satisfied:

- (1)  $S = \langle 1 \rangle$ ,
- (2)  $S$  is glued by completely glued semigroups.

If the embedding dimension of  $S$  is 2, then  $S$  is completely glued. If  $S$  is completely glued, then its type is one.

**Theorem 2.** If  $S$  is completely glued, then there is an RF-matrix of  $F(S)$  which satisfies (\*).

*Proof.* If  $S = \langle 1 \rangle$ , then the assertion is clear. Assume  $S = d_1 S_1 + d_2 S_2$  where  $d_1 \in S_2$ ,  $d_2 \in S_1$  and both  $S_1$  and  $S_2$  are completely glued. Then

$$F(S) = d_1 F(S_1) + d_2 F(S_2) + d_1 d_2$$

and there is an RF-matrix  $M_1$  (resp.  $M_2$ ) for  $F(S_1)$  (resp.  $F(S_2)$ ) in  $S_1$  (resp.  $(S_2)$ ) satisfying  $\det M_1 = (-1)^{s_1} F(S_1)$  (resp.  $\det M_2 = (-1)^{s_2} F(S_2)$ ) where  $s_1$  (resp.  $s_2$ ) is the embedding dimension of  $M_1$  (resp.  $M_2$ ). Since  $F(S_1) + d_2 \in S_1$  (resp.  $F(S_2) + d_1 \in S_2$ ), we may write  $F(S_1) + d_2 = \sum_i a_i n_i$  (resp.  $F(S_2) + d_1 = \sum_i a'_i n'_i$ ) where  $S_1 = \langle n_1, \dots, n_{s_1} \rangle$  (resp.  $S_2 = \langle n'_1, \dots, n'_{s_2} \rangle$ ) and  $a_i \geq 0$  (resp.  $a'_i \geq 0$ ) for each  $i$ . Let  $N_1$  (resp.  $N_2$ ) be an  $s_2 \times s_1$ -matrix (resp.  $s_1 \times s_2$ -matrix) whose  $ij$ -entry is  $a_i$  (resp.  $a'_i$ ) for each  $i, j$ . And put

$$M = \begin{pmatrix} M_1 & N_2 \\ N_1 & M_2 \end{pmatrix}.$$

Then  $M$  is an RF-matrix for  $F(S)$  in  $S$  and  $\det M = (-1)^{s_1+s_2} F(S)$ .  $\square$

**Theorem 3.** Assume  $s = 4$ . If the type of  $S$  is one, or if  $S$  is pseudo-symmetric, then there is an RF-matrix of  $F(S)$  which satisfies (\*). We say that  $S$  is *pseudo-symmetric* if  $\text{PF}(S) = \{F(S)/2, F(S)\}$ .

*Proof.* Let  $S = \langle n_1, n_2, n_3, n_4 \rangle$ . Assume  $t(S) = 1$ . Further, we may assume that  $S$  is not completely glued. Then, by [1], after suitable renumbering, we have

$$\text{RF}(F(S)) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{14} \\ a_{21} & -1 & \alpha_3 - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{32} & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 & a_{43} & -1 \end{pmatrix},$$

where  $\alpha_i$  is the minimal positive number satisfying that  $(\alpha_i - 1)n_i$  has the unique factorization by  $n_1, \dots, n_4$  for each  $i$  and  $0 < a_{21} < \alpha_1$ ,  $0 < a_{32} < \alpha_2$ ,  $0 < a_{43} < \alpha_3$ , and  $0 < a_{14} < \alpha_4$ . From this, we have  $\det \text{RF}(F(S)) = -F(S)$ .

Assume that  $S$  is pseudo-symmetric. By [3], after suitable renumbering, we also have

$$\begin{aligned} \text{RF}(F(S)) &= \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} - 1 & \alpha_3 - 1 & -1 \end{pmatrix}, \\ \text{RF}(F(S)/2) &= \begin{pmatrix} -1 & \alpha_2 - 1 & 0 & 0 \\ 0 & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} & 0 & -1 \end{pmatrix}, \end{aligned}$$

where  $\alpha_i$  is defined above and  $0 < a_{42} < \alpha_2$ . From this, we also have  $\det \text{RF}(F(S)) = -F(S)$ .  $\square$

Finally, we give some examples of RF-matrices in an almost symmetric semigroup.

**Definition.** Let  $S$  be a semigroup. For any  $f \in \text{PF}(S)$  with  $f \neq F(S)$ , if  $F(S) - f \in \text{PF}(S)$ , we say that  $S$  is *almost symmetric*.

**Example 3** (Watanabe's example). Let  $S = \langle 22, 46, 9, 57 \rangle$ . Then  $S$  is almost symmetric of type 3 and  $\text{PF}(S) = \{35, 70, 105\}$ . We also have

$$\text{RF}(70) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 8 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 9 & -1 \end{pmatrix}, \text{RF}(35) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 9 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}.$$

$$\det \text{RF}(70) = 0, \det \text{RF}(35) = -35.$$

**Example 4.** Let  $S = \langle 22, 26, 79, 83 \rangle$ . Then  $S$  is almost symmetric of type 3 and  $\text{PF}(S) = \{57, 238, 295\}$ . We also have

$$\text{RF}(57) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 5 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix}, \text{RF}(238) = \begin{pmatrix} -1 & 10 & 0 & 0 \\ 12 & -1 & 0 & 0 \\ 0 & 9 & -1 & 1 \\ 11 & 0 & 1 & -1 \end{pmatrix}$$

and

$$\text{RF}(295) = \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 0 & 8 & -1 & 2 \\ 10 & 0 & 2 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 4 & 11 & -1 & 0 \\ 10 & 0 & 2 & -1 \end{pmatrix}.$$

$\det \text{RF}(57) = \det \text{RF}(238) = 0$ . We note that the determinant of the former RF-matrix for 295 is zero, and that of the latter one is  $-295$ .

From above examples, it follows that the condition (\*) does not hold for all RF-matrices for pseudo-Frobenius numbers. Hence we modify the question as follows:

**Question.** Let  $S$  be a semigroup with embedding dimension  $s$ . Then, does the equation

$$(*) \quad \det \operatorname{RF}(F(S)) = (-1)^{s+1} F(S)$$

hold for some RF-matrix for  $F(S)$  ?

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